The Intrinsic Hodge Theory of Hyperbolic Curves

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I. Uniformization as a Species of Hodge Theory

(A.) Uniformizations Defined by the Exponential Function

In elementary mathematics, the simplest *variety* – i.e., geometric object defined by polynomial equations – that one encounters is a *line*, i.e., the set of points in the Euclidean plane $\mathbf{R}^2$ defined by an equation of the form $aX + bY = c$ (where $a, b, c \in \mathbf{R}$). After translation, rotation, and dilation, such an equation may be written in the form $X = 0$. In this case, the variety in question passes through the origin and, moreover, admits a natural *group structure*. Also, it is easy to understand in a very explicit way the totality of points $(x, y) \in \mathbf{R}^2$ that lie on this variety: Indeed, this set may be identified (via the projection $(x, y) \mapsto y$) with $\mathbf{R}$ itself.

The next simplest type of variety that one encounters is the variety in $\mathbf{R}^2$ defined by an equation of degree 2. After translation, rotation, reflection, and dilation of the X and Y coordinates, we see that such an equation is always one of the following three types: $X^2 + Y^2 = 1$, $XY = 1$, $X^2 = Y$. In the final case, $X^2 = Y$, the projection $(x, y) \mapsto x$ defines a natural bijection of the set of points on the variety with $\mathbf{R}$. In the first two cases, however, such projections do not give isomorphisms of the given variety with a linear variety.

In the first two cases, in order to find an explicit *cataloguing* indexed by $\mathbf{R}$ of all the points (x, y) lying on the variety – such a linear cataloguing is called a *uniformization* – it is necessary to introduce functions that are not algebraic, i.e., polynomial in nature. Namely, the maps

$$i \cdot t \mapsto e^{i \cdot t} \in \mathbf{C} = \{x + iy \mid x, y \in \mathbf{R}\} = \{(x, y) \mid x, y \in \mathbf{R}\} = \mathbf{R}^2$$

and

$$t \mapsto (e^t, e^{-t}) \in \mathbf{R}^2$$

where $t \in \mathbf{R}$, and e^t is the *exponential function*

$$e^t \stackrel{\text{def}}{=} 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} + \dots$$

1991 Mathematics Subject Classification: 14F30

Keywords and Phrases: hyperbolic curves, p-adic, fundamental group, Hodge theory

define such catalogues, or uniformizations, for the varieties $X^2 + Y^2 = 1$ and $XY = 1$, respectively. Moreover, these uniformizations are homomorphisms with respect to the additive group structure on $\mathbf{R}$ and certain natural group structures on the target varieties which *are* defined by polynomial equations. In the case of $XY = 1$, this uniformization map is a bijection, while in the case of $X^2 + Y^2 = 1$, the uniformization map is surjective, but has a kernel generated by $2\pi i \in \mathbf{R} \cdot i$. Since the uniformization map is normalized in both cases by the condition that the derivative at $0 \in \mathbf{R}$ have length 1, it thus follows that the number $2\pi i \in \mathbf{R} \cdot i$ is naturally associated to the unit circle (i.e., the variety defined by $X^2 + Y^2 = 1$). Such a number is referred to as a *period* of the variety.

(B.) Translation into the p-adic Case

Thus, in summary, in (A.) we saw that the exponential function allows us to give a natural and explicit accounting of all the $\mathbf{R}$-valued solutions of such equations as $X^2+Y^2 = 1$ and $XY = 1$. There are two natural ways to generalize the uniformization theory of (A.). The most obvious is to ask to what extent it may be generalized to more general types of equations, i.e., more general types of varieties. We will consider such questions in more detail in (C.), (D.) and (E.) below. Another natural way to attempt to generalize the theory of (A.) is to ask:

To what extent does the theory of (A.) extend to fields other than $\mathbf{R}$*?*

For instance, since the exponential function e^t is defined not only for $t \in \mathbf{R}$, but for all $t \in \mathbf{C}$, it is not difficult to see that that the theory of (A.) generalizes immediately to the field $\mathbf{C}$ of complex numbers. In fact, over $\mathbf{C}$, one sees that the varieties defined by $X^2 + Y^2 = 1$ and $XY = 1$ are actually the same variety. Thus, by working over $\mathbf{C}$, one sees that the theory of (A.) consists essentially of one case, not two.

Ultimately, the central motivation for wanting to generalize this uniformization theory to more general fields arises from *diophantine geometry*. Indeed,

Diophantine geometry is concerned precisely with the question of explicitly cataloguing all solutions of equations with values in a number field (i.e., finite extension field of $\mathbf{Q}$*).*

Thus, ideally, if one had a complete theory of uniformizations (= natural explicit catalogues of rational points of a variety) over an arbitrary number field, such as $\mathbf{Q}$, then (tautologically) one could solve the most central question of diophantine geometry, i.e., of cataloguing all rational solutions of polynomial equations.

At the present time, unfortunately, there does not yet exist such a theory of uniformizations over a number field. Nevertheless, if one has as one's goal the realization of such an *arithmetic uniformization theory*, it is natural to try to approximate this goal by first creating a theory of *uniformizations over p-adic fields*. For instance, if one has such a theory of p-adic uniformizations, one could try to concoct a global uniformization theory by somehow "gluing together" the local uniformization theories at the various infinite (i.e. "$\mathbf{R}$") and finite (i.e. p-adic) primes.

Let us begin by looking at the p-adic analogue of the theory of (A.). Since the two cases treated in (A.) are really one case (after one passes to an appropriate quadratic extension of the base field), let us, for simplicity, treat the case of the variety

$$\mathbf{G}_{\mathrm{m}} \stackrel{\mathrm{def}}{=} \mathrm{Spec}(\mathbf{Z}_p[X, X^{-1}])$$

(i.e., the variety defined by $XY = 1$). Note that $\mathbf{G}_{\mathrm{m}}$ has a natural group structure (just as in (A.)). Moreover, it has a natural *invariant differential* (so called because the differential is invariant with respect to translations arising from the group structure)

$$\omega \stackrel{\mathrm{def}}{=} \frac{dX}{X}$$

If one tries to construct the uniformization as a map from $\mathbf{G}_{\mathrm{m}}$ to some linear space (= the inverse of what we did in (A.) – i.e., the logarithm function), one sees that the desired uniformization should be obtained by somehow "integrating" this invariant differential $\omega = \frac{dX}{X}$. Thus, the question arises:

How does one effect such an "integration" of $\omega = \frac{dX}{X}$ in the p-adic context?

To understand how this works, let us first recall that in the classical case at the infinite prime, "integration" may be regarded as a *pairing between differentials and paths.* Moreover, since one's differentials are holomorphic, this pairing only depends on the *homotopy class* of a path. In the classical case, homotopy classes of (closed) paths make up the *fundamental group* of (the topological space underlying) the variety in question. Thus, in the p-adic context, it is natural to look for an analogue to integration in the form of a pairing between differentials and some sort of "p-adic fundamental group." What do we mean by "p-adic fundamental group"? In the classical case, the fundamental group may be thought of, equivalently, in terms of homotopy classes of paths as well as in terms of automorphism groups of coverings. In the p-adic case, one may then think of the "p-adic fundamental group" as the automorphisms of unramified Galois coverings of the variety in question whose Galois groups are (pro-)p-groups, i.e., (inverse limits of) finite groups of order a power of p.

In the case of $\mathbf{G}_{\mathrm{m}}$, the pairing in question is given (roughly speaking) as follows:

(1.) First, we pull back the differential ω via the covering

$$\phi_n : \mathbf{G}_{\mathrm{m}} \otimes_{\mathbf{Z}_p} \mathbf{Z}_p[\zeta_n] \longrightarrow \mathbf{G}_{\mathrm{m}} \otimes_{\mathbf{Z}_p} \mathbf{Z}_p[\zeta_n]$$

given by $X \mapsto X^{p^n}$ (where n is a positive integer, and ζ_n is a primitive p^n-th root of unity). Here, we think of the differential $\omega = \frac{dX}{X} \in \Omega_{\mathbf{G}_{\mathrm{m}}/\mathbf{Z}_p}$ as defining a differential $\omega_n \in \Omega_{\mathbf{G}_{\mathrm{m}} \otimes_{\mathbf{Z}_p} \mathbf{Z}_p[\zeta_n]/\mathbf{Z}_p}$.

(2.) Then, given an element γ of the Galois group of the above covering, we consider the difference

$$p^{-n}\{\gamma(\phi_n^*\omega_n) - \phi_n^*\omega_n\} \in \Omega_{\mathbf{Z}_p[\zeta_n]/\mathbf{Z}_p}$$

If one considers these differences over all n (as $n \to \infty$), one obtains a p-adic pairing:

$$(\mathbf{Z}_p \cdot \omega) \otimes \pi_1^{(p)}((\mathbf{G}_\mathrm{m})_{\overline{\mathbf{Q}}_p}) \to \mathbf{C}_p(1)$$

Here, $\overline{\mathbf{Q}}_p$ is an algebraic closure of $\mathbf{Q}_p$; $\mathbf{C}_p$ is the p-adic completion of $\overline{\mathbf{Q}}_p$; and the "(1)" following the $\mathbf{C}_p$ is a "Tate twist," i.e., if we think of all the objects involved as being equipped with a natural action of $\Gamma_{\mathbf{Q}_p} \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$, then the "(1)" means that the given action (e.g., the natural action on $\mathbf{C}_p$) is twisted by the action of the cyclotomic character $\Gamma_{\mathbf{Z}_p} \to {\mathbf{Z}_p}^\times$. Finally, $\pi_1^{(p)}((\mathbf{G}_\mathrm{m})_{\overline{\mathbf{Q}}_p})$ is the (geometric) "p-adic fundamental group" of $\mathbf{G}_\mathrm{m}$ discussed above. Since essentially all the coverings that make up this fundamental group are given as in (1.) above, we see immediately that $\pi_1^{(p)}((\mathbf{G}_\mathrm{m})_{\overline{\mathbf{Q}}_p})$ may be identified with $\mathbf{Z}_p(1)$.

Then the fundamental *theorem* here (analogous to the uniformization theory of (A.)) is that this pairing defines an isomorphism

$$\mathbf{C}_p \cdot \omega \cong H^1_{\mathrm{et}}((\mathbf{G}_\mathrm{m})_{\overline{\mathbf{Q}}_p}, \mathbf{C}_p(1)) \stackrel{\mathrm{def}}{=} \mathrm{Hom}_{\mathbf{Z}_p}(\pi_1^{(p)}((\mathbf{G}_\mathrm{m})_{\overline{\mathbf{Q}}_p}), \mathbf{C}_p(1))$$

(where the "et" stands for "étale cohomology"). Less rigorously, but more intuitively, this theorem asserts an equivalence

$$\frac{dX}{X} \longleftrightarrow \{X^{1/p^n}\}_{n \in \mathbf{N}}$$

Here, the right-hand side denotes the "coverings of $\mathbf{G}_\mathrm{m}$ defined by taking p-power roots of X." Note that both sides are a sort of *representation of the logarithm function* – which is natural since the logarithm/exponential function is fundamental to the theory of (A.). Namely, the left-hand side is "the derivative of the logarithm of X" while the right-hand side, at least from a "real/complex analytic point of view" is

$$1 + \frac{1}{p^n}\log(X) + \ldots$$

In fact, in the p-adic case, one can define a natural logarithm function $\log(X)$ which converges in a small p-adic neighborhood of $1 \in \mathbf{G}_\mathrm{m}(\mathbf{Z}_p)$ via the same power series in

$1-X$ as in the complex case. Then the derivative of this function will be equal to ω, and $X^{\frac{1}{p^n}}$ can be written as the same power series in $\log(X)$ as in the complex case. Moreover, the inverse to this power series also converges in a sufficiently small p-adic neighborhood $(\mathbf{C}_p)_{\leq \epsilon} \subseteq \mathbf{C}_p$ of $0 \in \mathbf{C}_p$ and defines a "uniformization map":

$$\exp_{\mathbf{G}_\mathrm{m}} : (\mathbf{C}_p)_{\leq \epsilon} \to \mathbf{G}_\mathrm{m}(\mathbf{C}_p)$$

Yet another way to regard this fundamental isomorphism is relative to the above discussion of integrating $\omega = \frac{dX}{X}$ along (closed) paths. Since we are dealing with the fundamental group here, our paths are, in fact, closed, and so, from the discussion of (A.), we should expect to get the analogue of integrating ω along closed paths, i.e., multiples of the period $2\pi i$. Thus, the above isomorphism may also be regarded as a sort "p-adic period," or "p-adic version of $2\pi i$." This is precisely the point of view of *p-adic Hodge theory*. Note further that in the analytic case, if we consider the logarithm function $\log(X)$ on the universal covering of $(\mathbf{G}_\mathrm{m})_\mathbf{C} \stackrel{\mathrm{def}}{=} \mathrm{Spec}(\mathbf{C}[X, X^{-1}])$, and let γ be a deck transformation of this universal covering, then $\gamma(\log(X)) - \log(X)$ is a multiple of the period $2\pi i$. This expression $\gamma(\log(X)) - \log(X)$ is reminiscent of the recipe given above (cf. (2.)) for the p-adic period map.

Finally, before proceeding, we note that if instead of using "$\mathbf{C}_p$" coefficients, one takes as one's coefficients a sort of very large ring of p-adic functions "$\overline{R}^\wedge_{\mathbf{Q}_p}$," then one can prove a similar isomorphism

$$\overline{R}^\wedge_{\mathbf{Q}_p} \otimes_{\mathcal{O}_U} \Omega_{U/\mathbf{Z}_p} \cong H^1_{\mathrm{et}}(U, \overline{R}^\wedge_{\mathbf{Q}_p}(1))$$

for any sufficiently small open U of any smooth one-dimensional (p-adic formal) scheme X over $\mathbf{Z}_p$. This is the content of Faltings' *theory of almost étale extensions* and forms the basis of Faltings' *p-adic Hodge theory* ([Falt2]).

(C.) Elliptic Curves

Now we come to the question of generalizing the above uniformization theory to more general varieties. So far, we have considered the case of varieties in the plane defined by equations of degree 1 and 2. The next case, then, to consider is the case of a variety in the plane defined by an equation of degree 3. Up to various elementary algebraic operations (such as adding the point at infinity), this case is essentially the case of *elliptic curves*, i.e., (in the language of schemes) smooth, proper, geometrically connected curves of genus 1.

Let E be an elliptic curve over $\mathbf{C}$. Write $T(E)$ for the tangent space to E at its origin. Thus, $T(E)$ is a one-dimensional complex vector space. Then E admits a natural *exponential map*:

$$\exp_E : T(E) \to E$$

which *uniformizes* E. If one chooses an appropriate basis of $T(E)$, then this uniformization allows one to write E in the form

$$\mathbf{C}/\Lambda$$

where $\Lambda \stackrel{\text{def}}{=} \mathbf{Z} + \mathbf{Z} \cdot \tau$, for some $\tau \in \mathfrak{H} \stackrel{\text{def}}{=} \{z \in \mathbf{C} | \ \mathrm{Im}(z) > 0\}$. Moreover, we observe that this uniformization of E also *induces a uniformization of the moduli space* (actually, an algebraic stack) *of elliptic curves.* Indeed, since elliptic curves are algebraic objects, this moduli space $\mathcal{M}_{1,0}$ also admits a natural structure of algebraic variety. Thus, the correspondence

$$\tau \ (\in \mathfrak{H}) \ \mapsto \mathbf{C}/(\mathbf{Z} + \mathbf{Z} \cdot \tau)$$

defines a *uniformization map*

$$\mathfrak{H} \to \mathcal{M}_{1,0}$$

This map induces an isomorphism between the upper half-plane $\mathfrak{H}$ and the universal covering space of $\mathcal{M}_{1,0}$ (thought of as a stack). Put another way, this map gives us a natural coordinate – namely, τ – on (the universal cover of) $\mathcal{M}_{1,0}$ which gives us an *explicit catalogue* of all the $\mathbf{C}$-valued points $\mathcal{M}_{1,0}(\mathbf{C})$, i.e., all the $\mathbf{C}$-valued solutions of the polynomial equations defining the algebraic object $\mathcal{M}_{1,0}$.

Next, we observe that the above complex theory has a well-known p-adic analogue. Namely, just as in (B.) above, we considered the p-adic Hodge theory of $\mathbf{G}_{\mathrm{m}}$, one may also consider the *p-adic Hodge theory of elliptic curves.* In the case of elliptic curves, however, the field $\mathbf{C}_p$ is not sufficient to define the "p-adic periods." Instead, one must work with bigger rings, such as the ring B_{crys} of Fontaine. Let E be an elliptic curve over a finite extension K of $\mathbf{Q}_p$. For simplicity, let us assume that E admits an extension to an elliptic curve over the ring of integers $\mathcal{O}_K$. One then obtains an isomorphism (after tensoring up to B_{crys})

$$H^1_{\mathrm{DR}}(E, \mathcal{O}_E) \otimes_K B_{\mathrm{crys}} \cong H^1_{\mathrm{et}}(E, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} B_{\mathrm{crys}}$$

between the de Rham and étale cohomologies (in dimension one) of the elliptic curve. Here the de Rham cohomology $H^1(E, \mathcal{O}_E)$ of E may be thought of as the analogue for an elliptic curve of the one-dimensional space generated by the invariant differential $\frac{dX}{X}$ in the case of $\mathbf{G}_{\mathrm{m}}$. The étale cohomology $H^1_{\mathrm{et}}(E, \mathbf{Q}_p)$ may be thought of as

$$\mathrm{Hom}_{\mathbf{Z}_p}(\pi_1^{(p)}(E_{\overline{K}}), \mathbf{Q}_p)$$

where $E_{\overline{K}} \stackrel{\text{def}}{=} E \otimes_K \overline{K}$; $\overline{K}$ is an algebraic closure of K; and $\pi_1^{(p)}(E_{\overline{K}})$ is the "geometric p-adic fundamental group of E." This étale cohomology module $H^1_{\mathrm{et}}(E, \mathbf{Q}_p)$ is equipped with a natural action of the Galois group $\Gamma_K \stackrel{\text{def}}{=} \text{Gal}(\overline{K}/K)$. Thus, the above *comparison isomorphism* may be regarded as asserting an equivalence between the de Rham cohomology (equipped with certain natural auxiliary structures like the Hodge filtration and Frobenius action) and the étale cohomology (equipped with its Galois action) of E.

The p-adic Hodge theory of an elliptic curve is easiest to understand in the *ordinary case*. This is the case where the reduction modulo $\mathfrak{m}_K$ (= the maximal ideal of $\mathcal{O}_K$) of the elliptic curve has a nonzero Hasse invariant. Another equivalent characterization is that the Cartier operator on the invariant differentials of the reduction modulo p is bijective. It turns out that in some sense, "most" elliptic curves are ordinary. In the ordinary case, the $\Gamma_K \stackrel{\text{def}}{=} \text{Gal}(\overline{K}/K)$-module $M \stackrel{\text{def}}{=} H^1_{\text{et}}(E, \mathbf{Z}_p(1))$ fits into an exact sequence of Γ_K-modules

$$0 \to M_1 \to M \to M_0 \to 0$$

where M_1 and M_0 are $\mathbf{Z}_p$-modules of rank 1. If one then restricts from K to the p-adic completion $\widehat{K}^{\text{unr}}$ of the maximal unramified extension of K, then the following phenomena occur:

(1.) Over $\widehat{K}^{\text{unr}}$, E becomes isomorphic, in a formal neighborhood of the origin, to $\mathbf{G}_{\text{m}}$. This isomorphism respects the group structures on E and $\mathbf{G}_{\text{m}}$.

(2.) The resulting $\Gamma_{\widehat{K}^{\text{unr}}}$-modules M_1 and M_0 become isomorphic to $\mathbf{Z}_p(1)$ and $\mathbf{Z}_p$, respectively. The above exact sequence then defines an extension class

$$q_E \in H^1_{\text{et}}(\widehat{K}^{\text{unr}}, \mathbf{Z}_p(1)) \cong \{(\widehat{K}^{\text{unr}})^\times\}^\wedge$$

(where the "$\cong$" is given by Kummer theory). In fact, this class $q_E \in (\mathcal{O}^\times_{\widehat{K}^{\text{unr}}})^\wedge$. Moreover, the assignment

$$E \mapsto q_E \in (\mathcal{O}^\times_{\widehat{K}^{\text{unr}}})^\wedge$$

defines a *bijection* between all deformations of the reduction modulo $\mathfrak{m}_{\widehat{K}^{\text{unr}}}$ of E to an elliptic curve over $\mathcal{O}_{\widehat{K}^{\text{unr}}}$ and the set $(\mathcal{O}^\times_{\widehat{K}^{\text{unr}}})^\wedge$.

In other words, (by composing the isomorphism of (1.) above with $\exp_{\mathbf{G}_{\text{m}}}$) one may regard (1.) as a sort of p-adic analogue of the *uniformization of an elliptic curve over the complex numbers*, while the bijection of (2.) (which is in some sense induced by "pushing forward" the uniformization of (1.)) may be regarded as a local p-adic uniformization of the moduli space $\mathcal{M}_{1,0}$ of elliptic curves, hence as a sort of p-adic analogue of the *uniformization of $\mathcal{M}_{1,0}$ (over $\mathbf{C}$) by the upper half-plane $\mathfrak{H}$*. The bijection of (2.) is often referred to as *Serre-Tate theory for ordinary elliptic curves*.

(D.) The Notion of Intrinsic Hodge Theory

Next, we would like to see to what extent the theory discussed so far may be generalized to plane curves of degree > 3, and, more generally, to arbitrary varieties. Before we discuss this generalization, however, we must consider in more detail precisely what we would like to generalize. That is to say, because the examples we have considered so far are relatively simple, they have many different aspects. Thus, when one wishes to consider generalizations, one must specify which aspects one wants the generalizations to retain and which aspects one is willing to sacrifice as "inessential" when one generalizes.

One central aspect of both the complex and p-adic theories that we have discussed so far is that they involve some sort of *comparison* or *equivalence* between what one might call the "de Rham world" (of polynomials and their differentials) and the "topology/geometry world" (over $\mathbf{C}$) or the "p-adic étale topology plus Galois action world" (in the p-adic case). This sort of equivalence, or comparison isomorphism,

algebraic geometry $\Longleftrightarrow$

topology+ (differential) geometry/p-adic étale topology plus Galois action

is the main theme of (both complex and p-adic) *Hodge theory.* The most basic example of such a comparison isomorphism, or Hodge theory, is (in the complex case) the de Rham isomorphism between de Rham and singular cohomology, or (in the p-adic case) the isomorphism (over such rings as B_{crys}) between the algebraic de Rham cohomology and the p-adic étale cohomology (with Galois action). This *Hodge theory of cohomologies* is thus a direct generalization of the comparison isomorphisms discussed above in (B.) and (C.). In fact, although there now exist a number of proofs of the main theorems of p-adic Hodge theory, one proof (due to Faltings – [Falt3]) is based precisely on the local isomorphism

$$\overline{R}^{\wedge}_{\mathbf{Q}_p} \otimes_{\mathcal{O}_U} \Omega_{U/\mathbf{Z}_p} \cong H^1_{\mathrm{et}}(U, \overline{R}^{\wedge}_{\mathbf{Q}_p}(1))$$

discussed at the end of (B.). Namely, the comparison isomorphism in the general case (for varieties of arbitrary dimension) is obtained by essentially gluing together products of the above local isomorphism applied to sufficiently small opens of a given variety over a p-adic field.

In the present discussion, however, we wish to consider a different type of generalization. That is to say, although we still want the generalization to be some sort of Hodge theory, i.e., some sort of equivalence as discussed above, we would like to consider Hodge theories for which the object appearing on the algebraic geometry side is (not some cohomology module associated to the given variety, but) the *"variety itself,"* i.e., the rational points or moduli of the given variety. That is to say, although it just so happened that in the case of relatively simple varieties like $\mathbf{G}_{\mathrm{m}}$ or elliptic curves, the "variety itself" happened to be *essentially embodied* in its first cohomology module (in more fancy language: elliptic curves and $\mathbf{G}_{\mathrm{m}}$ *"are"* 1-motives), in general, the cohomology modules of a given variety only embody certain limited aspects of the variety, and are not "essentially equivalent" to the "variety itself." In the following, we shall refer to Hodge theories of the "variety itself" as *intrinsic Hodge theories*, or IHT's for short (to be distinguished from the more classical "Hodge theories of cohomologies"). Those intrinsic Hodge theories for which the expression "variety itself" turns out to mean "the rational points of the variety" will be called *physical* (since they give a new way to "physically recover the variety"), while those intrinsic Hodge theories for which the expression "variety itself" is used to mean "the moduli of the variety" will be called *modular*.

If one thinks back to our discussion of uniformizations of algebraic varieties as *natural "geometric"* (our translation for "linear" in the general case) *catalogues of all rational points*, one sees that it is essentially a tautology of terminology that

$$\text{physical IHT} \iff \text{uniformization theory of the given variety}$$

$$\text{modular IHT} \iff \text{uniformization theory of the moduli space of the given variety}$$

Nonetheless, although we will be concerned with a different sort of generalization of the theory of (A.), (B.), and (C.) from the classical Hodge theory of cohomologies, the techniques that will be employed in the proofs of the main theorems that we will discuss in II. and III. below will be based on the techniques of (cohomological) p-adic Hodge theory (as in [Falt2,3]).

(E.) Curves of Higher Genus: the Fuchsian Uniformization

Finally, we come to the case of plane curves of higher (i.e., > 1) genus, or, more generally, hyperbolic curves. A *hyperbolic curve* is an algebraic curve obtained by removing r points from a smooth, proper curve of genus g, where g and r are nonnegative integers such that $2g - 2 + r > 0$. If X is a hyperbolic curve over the field of complex numbers $\mathbf{C}$, then X gives rise in a natural way to a Riemann surface $\mathcal{X}$. As one knows from complex analysis, the most fundamental fact concerning such a Riemann surface (due to Köbe) is that it may be *uniformized by the upper half-plane*, i.e.,

$$\mathcal{X} \cong \mathfrak{H}/\Gamma$$

where $\mathfrak{H} \stackrel{\text{def}}{=} \{z \in \mathbf{C} \mid \text{Im}(z) > 0\}$, and $\Gamma \cong \pi_1(\mathcal{X})$ (the topological fundamental group of $\mathcal{X}$) is a discontinuous group acting on $\mathfrak{H}$. This uniformization is referred to as the *Fuchsian uniformization of* $\mathcal{X}$. Thus, one sees immediately that the Fuchsian uniformization, i.e.,

$$X(\mathbf{C}) \cong \mathfrak{H}/\Gamma$$

may be regarded as being a *physical IHT of the hyperbolic curve* X.

This is not the only way to interpret the Fuchsian uniformization, however. That is to say, it also has a *modular aspect*, as follows. First, note that the action of Γ on $\mathfrak{H}$ defines a *canonical representation*

$$\rho_{\mathcal{X}} : \pi_1(\mathcal{X}) \to PSL_2(\mathbf{R}) \stackrel{\text{def}}{=} SL_2(\mathbf{R})/\{\pm 1\} = \text{Aut}_{\text{Holomorphic}}(\mathfrak{H})$$

Note that $\rho_{\mathcal{X}}$ may also be regarded as a representation into $PGL_2(\mathbf{C}) = GL_2(\mathbf{C})/\mathbf{C}^{\times}$, hence as defining an action of $\pi_1(\mathcal{X})$ on $\mathbf{P}^1_{\mathbf{C}}$. Taking the quotient of $\mathfrak{H} \times \mathbf{P}^1_{\mathbf{C}}$ by the action of $\pi_1(\mathcal{X})$ on both factors then gives rise to a projective bundle with connection on $\mathcal{X}$. It is immediate that this projective bundle and connection may be algebraized, so we thus obtain a projective bundle and connection $(P \to X, \nabla_P)$ on X. This pair (P, ∇_P) has certain properties which make it an *indigenous bundle* (terminology due to Gunning). More generally, an indigenous bundle on $\mathcal{X}$ may be thought of as a *projective structure* on $\mathcal{X}$, i.e., a subsheaf of the sheaf of holomorphic functions on $\mathcal{X}$ such that locally any two sections of this subsheaf are related by a linear fractional transformation. Thus, the Fuchsian uniformization defines a special *canonical indigenous bundle* on X.

In fact, the notion of an indigenous bundle is entirely *algebraic*. Thus, one has a natural moduli stack $\mathcal{S}_{g,r} \to \mathcal{M}_{g,r}$ of indigenous bundles, which forms a torsor (under the affine group given by the sheaf of differentials on $\mathcal{M}_{g,r}$) – called the *Schwarz torsor* – over the *moduli stack* $\mathcal{M}_{g,r}$ of hyperbolic curves of type (g, r). Moreover, $\mathcal{S}_{g,r}$ is not only algebraic, it is defined over $\mathbf{Z}[\frac{1}{2}]$. Thus, the canonical indigenous bundle defines a *canonical real analytic section*

$$s : \mathcal{M}_{g,r}(\mathbf{C}) \to \mathcal{S}_{g,r}(\mathbf{C})$$

of the Schwarz torsor at the infinite prime. Moreover, not only does s "contain" all the information that one needs to define the Fuchsian uniformization of an individual hyperbolic curve (indeed, this much is obvious from the definition of s!), it also essentially "is" (interpreted properly) the *Bers uniformization* of the universal covering space (i.e., "Teichmüller space") of $\mathcal{M}_{g,r}(\mathbf{C})$ (cf. the discussions in the Introductions of [Mzk1,4] for more details). That is to say, the study of this canonical section s may be regarded as the realization of the Fuchsian uniformization as a *modular IHT*. Alternatively, from the point of view of classical Teichmüller theory, one may regard the uniformization theory of the moduli of hyperbolic curves as the theory of (so-called) *quasi-fuchsian deformations of the representation* $\rho_{\mathcal{X}}$.

In II. and III. below, we wish to discuss *p-adic analogues* of the above physical and modular IHT's for complex hyperbolic curves.

At this point, the reader might ask if we can continue in this fashion to obtain IHT's for arbitrary algebraic varieties. Unfortunately, however, even over $\mathbf{C}$ (where things tend to be much better understood than in the p-adic case), the uniformization and moduli theory of algebraic varieties of dimension ≥ 2 is far from being well-understood. Thus, in the present manuscript, we shall restrict our attention to curves.

II. The Anabelian Geometry of Hyperbolic Curves

(A.) Grothendieck's Anabelian Philosophy

Let K be a field of characteristic zero. Let us denote by $\overline{K}$ an algebraic closure of K. Let $\Gamma_K \stackrel{\text{def}}{=} \text{Gal}(\overline{K}/K)$. Suppose that X_K is a *variety over* K. Then we will denote by

$$\pi_1(X_K)$$

the *algebraic fundamental group of* X_K (cf. [SGA1]). This group is a compact, profinite topological group, well-defined up to inner automorphism (since we did not specify a "base-point"), and which has the following property: The category of finite sets with a continuous $\pi_1(X_K)$-action is naturally equivalent to the category of finite étale coverings of X_K. Moreover, if K is, for instance, an algebraically closed subfield of $\mathbf{C}$, then $\pi_1(X_K)$ may be identified with the *profinite completion* (= inverse limit of all finite quotients) *of the usual topological fundamental group* $\pi_1^{\text{top}}(X_{\mathbf{C}})$ (where $X_{\mathbf{C}} \stackrel{\text{def}}{=} X_K \otimes_K \mathbf{C}$).

Now let X_K be a *hyperbolic curve over* K; write $X_{\overline{K}} \stackrel{\text{def}}{=} X \times_K \overline{K}$. Then one has an exact sequence

$$1 \to \pi_1(X_{\overline{K}}) \to \pi_1(X_K) \to \Gamma_K \to 1$$

of *algebraic fundamental groups.* We shall refer to $\pi_1(X_{\overline{K}})$ as the *geometric fundamental group* of X_K. Note that, by the above discussion of the case where $\overline{K} \subseteq \mathbf{C}$, it follows that the structure of $\pi_1(X_{\overline{K}})$ is determined entirely by (g, r) (i.e., the "type" of the hyperbolic curve X_K). In particular, $\pi_1(X_{\overline{K}})$ *does not depend on the moduli of* X_K. Of course, this results from the fact that K is of *characteristic zero.* In positive characteristic, on the other hand, preliminary evidence ([Tama2]) suggests that the fundamental group of a hyperbolic curve over an algebraically closed field (far from being independent of the moduli of the curve!) may in fact *completely determine* the moduli of the curve.

We shall refer to $\pi_1(X_K)$ (equipped with its augmentation to Γ_K) as the *arithmetic fundamental group* of X_K. Although it is made up of two "parts" – i.e., $\pi_1(X_{\overline{K}})$ and Γ_K – which do not depend on the moduli of X_K, it is not unreasonable to expect that the extension class defined by the above exact sequence, i.e., the structure of $\pi_1(X_K)$ as a group equipped with augmentation to Γ_K, may in fact depend quite strongly on the moduli of X_K. Indeed, according to the *anabelian philosophy* of Grothendieck (cf. [LS]), for "sufficiently arithmetic" K, one expects that *the structure of the arithmetic fundamental group* $\pi_1(X_K)$ *should be enough to determine the moduli of* X_K. Although many important versions of Grothendieck's anabelian conjectures remain unsolved (most notably the so-called *Section Conjecture* (cf., e.g., [LS], p. 289, 2)), in the remainder of this §, we shall discuss various versions that have been resolved in the affirmative. For instance, such a version of these conjectures which will be discussed in (B.) below (Theorem 1) states roughly that *(nonconstant) morphisms from a smooth K-variety to X_K should be in bijective correspondence with (open) homomorphisms (over Γ_K) between the corresponding arithmetic fundamental groups.* Thus, there is an obvious analogy between this (form of Grothendieck's) conjecture and the Tate conjecture on abelian varieties, which states roughly that morphisms between abelian varieties are equivalent to morphisms between their arithmetic fundamental groups.

Note that this anabelian philosophy is a *special case of the notion of "intrinsic Hodge theory"* discussed above: indeed, on the algebraic geometry side, one has "the curve itself," whereas on the topology plus arithmetic side, one has the arithmetic fundamental group, i.e., the purely (étale) topological $\pi_1(X_{\overline{K}})$, equipped with the structure of extension given by the above exact sequence.

In fact, it is interesting to note – especially relative to the discussion at the beginning of I., (B.), above – that Grothendieck's anabelian philosophy arose as an *approach to diophantine geometry*. It is primarily for this reason that it was originally thought that the most natural sort of "arithmetic" base field K over which one should expect Grothendieck's anabelian conjectures to hold was a *number field*. Another reason for the idea that the base field in these conjectures should be a number field was the analogy with Tate's conjecture on homomorphisms between abelian varieties (cf., e.g., [Falt1]). Indeed, in discussions of Grothendieck's anabelian philosophy, it was common to refer to statements such as that of Theorem 1 below as the "anabelian Tate conjecture," or the "Tate conjecture for hyperbolic curves." In fact, however, there is an important difference between Theorem 1 and the "Tate conjecture" of, say, [Falt1]: Namely, whereas Theorem 1 below holds over *local fields* (i.e., finite extensions of $\mathbf{Q}_p$), *the Tate conjecture for abelian varieties is false over local fields.* Moreover, until the proof of Theorem 1, it was generally thought that, just like its abelian cousin, the "anabelian Tate conjecture" was essentially global in nature. That is to say, it appears that the point of view of the author, i.e., that Theorem 1 should be regarded as a p-adic version of the "physical aspect" of the Fuchsian uniformization of a hyperbolic curve, does not exist in the literature (prior to the work of the author).

(B.) The Main Result

Building on earlier work of H. Nakamura and A. Tamagawa (see, especially, [Tama1]), the author applied the p-adic Hodge theory of [Falt2] and [BK] to prove the following result (cf. Theorem A of [Mzk5]):

Theorem 1. *Let p be a prime number. Let K be a subfield of a finitely generated field extension of $\mathbf{Q}_p$. Let X_K be a hyperbolic curve over K. Then for any smooth variety S_K over K, the natural map*

$$X_K(S_K)^{\mathrm{dom}} \to \mathrm{Hom}^{\mathrm{open}}_{\Gamma_K}(\pi_1(S_K), \pi_1(X_K))$$

is bijective. Here, the superscripted "dom" denotes dominant ($\iff$ nonconstant) K-morphisms, while $\mathrm{Hom}^{\mathrm{open}}_{\Gamma_K}$ denotes open, continuous homomorphisms compatible with the *augmentations to* Γ_K*, and considered up to composition with an inner automorphism arising from* $\pi_1(X_{\overline{K}})$.

Note that this result constitutes an analogue of the "physical aspect" of the Fuchsian uniformization, i.e., it exhibits the *scheme* X_K (in the sense of the functor defined by considering (nonconstant) K-morphisms from arbitrary smooth S_K to X_K) as equivalent to the "physical/analytic object"

$$\mathrm{Hom}^{\mathrm{open}}_{\Gamma_K}(-, \pi_1(X_K))$$

defined by the topological $\pi_1(X_{\overline{K}})$ together with some additional canonical arithmetic structure (i.e., $\pi_1(X_K)$).

In fact, various slightly stronger versions of Theorem 1 hold. For instance, instead of the whole geometric fundamental group $\pi_1(X_{\overline{K}})$, it suffices to consider its maximal pro-p quotient $\pi_1(X_{\overline{K}})^{(p)}$. Indeed, this "$\pi_1(X_{\overline{K}})^{(p)}$" is the precise definition of what was meant by the expression "*p-adic fundamental group*" in the discussion of I., (B.). Another strengthening allows one to prove the following result (cf. Theorem B of [Mzk5]), which generalizes a result of Pop ([Pop]):

Theorem 2. *Let* p *be a prime number. Let* K *be a subfield of a finitely generated field extension of* $\mathbf{Q}_p$*. Let* L *and* M *be function fields of arbitrary dimension over* K*. Then the natural map*

$$\mathrm{Hom}_K(\mathrm{Spec}(L), \mathrm{Spec}(M)) \to \mathrm{Hom}^{\mathrm{open}}_{\Gamma_K}(\Gamma_L, \Gamma_M)$$

is bijective. Here, $\mathrm{Hom}^{\mathrm{open}}_{\Gamma_K}(\Gamma_L, \Gamma_M)$ *is the set of open, continuous group homomorphisms* $\Gamma_L \to \Gamma_M$ *over* Γ_K*, considered up to composition with an inner homomorphism arising from* $\mathrm{Ker}(\Gamma_M \to \Gamma_K)$.

(C.) The Idea of the Proof

To understand the proof of Theorem 1, let us first recall the situation in the complex case. If X is a hyperbolic curve over $\mathbf{C}$, then as discussed in I., (E.), we have an isomorphism

$$X(\mathbf{C}) \cong \mathfrak{H}/\Gamma$$

One way to think of this isomorphism is as the datum of an *algebraic structure* (i.e., the structure of X as an *algebraic* curve) on the *analytic quotient* $\mathfrak{H}/\Gamma$. In complex analysis, this sort of algebraic structure is typically constructed by starting with the upper half-plane $\mathfrak{H}$, and then constructing various *automorphic forms on* $\mathfrak{H}$, i.e., differential forms on

$\mathfrak{H}$ that are invariant with respect to the natural action of Γ. These differential forms then define a morphism

$$\mathfrak{H} \to \mathbf{P}$$

from $\mathfrak{H}$ to some projective space $\mathbf{P}$ whose image is precisely the algebraic curve X. The point here is that although the automorphic forms that one constructs on $\mathfrak{H}$ will eventially be seen to be "algebraic" (i.e., as differential forms on X), their construction (as differentials forms on $\mathfrak{H}$) is *entirely analytic*. Note that this technique of constructing an algebraic structure on an analytic quotient appears frequently in the theory of *Shimura varieties* (where the automorphic forms are Poincaré series, Eisenstein series, etc.).

It is precisely this analytic argument that was the motivating idea behind the p-adic proof of Theorem 1. Indeed, suppose that one is given two *hyperbolic curves* X_K, Y_K over K. For simplicity, let us assume that both X_K and Y_K are both *proper* and *non-hyperelliptic*, and that K is a *finite* extension of $\mathbf{Q}_p$. Suppose, moreover, that we are given an isomorphism

$$\alpha : \pi_1(X_K) \cong \pi_1(Y_K)$$

of the respective arithmetic fundamental groups which is compatible with the projections to Γ_K. Then Theorem 1 states that α necessarily *arises geometrically*, i.e., from a K-isomorphism $X_K \cong Y_K$. In the following, we would like to give a rough sketch of the ideas used to prove this result.

First, observe that α induces an isomorphism

$$\pi_1^{(p)}(X_{\overline{K}})^{\mathrm{ab}} \cong \pi_1^{(p)}(Y_{\overline{K}})^{\mathrm{ab}}$$

between the abelianizations of the maximal pro-p quotients of the respective geometric fundamental groups. Then it follows from p-adic Hodge theory that if one tensors this isomorphism with $\mathbf{C}_p$ (i.e, the p-adic completion of $\overline{K}$), and then takes Γ_K-invariants, one obtains (naturally) on both sides the respective spaces of global differentials, $D_X \stackrel{\mathrm{def}}{=} H^0(X_K, \omega_{X_K})$ and $D_Y \stackrel{\mathrm{def}}{=} H^0(Y_K, \omega_{Y_K})$. Thus, one obtains an isomorphism

$$D_X \cong D_Y$$

induced by α. Let $P_X \stackrel{\mathrm{def}}{=} \mathbf{P}(D_X)$, $P_Y \stackrel{\mathrm{def}}{=} \mathbf{P}(D_Y)$ be the corresponding projective spaces. Thus, one obtains an isomorphism $P_X \cong P_Y$. On the other hand, since we assumed that X_K and Y_K are *non-hyperelliptic*, it follows from elementary algebraic geometry that we have canonical embeddings $X_K \subseteq P_X$, $Y_K \subseteq P_Y$. In other words, we have a diagram:

$$\begin{array}{ccc} P_X & \cong & P_Y \\ \bigcup & & \bigcup \\ X_K & \stackrel{?}{\longrightarrow} & Y_K \end{array}$$

Thus, the problem of constructing an isomorphism $X_K \cong Y_K$ as desired is reduced to showing that the isomorphism $P_X \cong P_Y$ that we have already constructed maps X_K into Y_K. This is proven precisely by considering certain *p-adic analytic representations* of the differentials of D_X and D_Y as differentials on a certain p-adic space (= the spectrum of a certain large p-adic field) in a fashion reminiscent of the way in which analytic representations (i.e., automorphic forms) of differential forms appeared in the above discussion of the complex case. We refer to [Mzk5], [NTM], for more details.

III. Teichmüller Theory over the p-adics

(A.) The Motivating Example: Shimura Curves

As discussed in I., (E.), classical complex Teichmüller theory may be formulated as the study of the canonical real analytic section s of the Schwarz torsor $\mathcal{S}_{g,r} \to \mathcal{M}_{g,r}$. Thus, it is natural to suppose that the p-adic analogue of classical Teichmüller theory should revolve around some sort of *canonical p-adic section* of the Schwarz torsor. Then the question arises:

How does one define a canonical p-adic section of the Schwarz torsor?

Put another way, for each (or at least most) p-adic hyperbolic curves, we would like to associate a (or at least a finite, bounded number of) canonical indigenous bundles. Thus, we would like to know what sort of properties such a "canonical indigenous bundle" should have.

The model that provides the answer to this question is the theory of *Shimura curves*. In fact, the theory of canonical Schwarz structures, canonical differentials, and canonical coordinates on Shimura curves localized at finite primes has been extensively studied by Y. Ihara (see, e.g., [Ihara]). In some sense, *Ihara's theory provides the prototype for the "p-adic Teichmüller theory" of arbitrary hyperbolic curves* ([Mzk1-4]) to be discussed in (B.) and (C.) below. The easiest example of a Shimura curve is $\mathcal{M}_{1,0}$, the moduli stack of elliptic curves. In this case, the projectivization of the rank two bundle on $\mathcal{M}_{1,0}$ defined by the first de Rham cohomology module of the universal elliptic curve on $\mathcal{M}_{1,0}$ gives rise (when equipped with the Gauss-Manin connection) to the canonical indigenous bundle on $\mathcal{M}_{1,0}$. Moreover, it is well-known that the *p-curvature* (a canonical invariant of bundles with connection in positive characteristic which measures the extent to which the connection is compatible with Frobenius) of this bundle has the following property:

The p-curvature of the canonical indigenous bundle on $\mathcal{M}_{1,0}$ (reduced mod p) is square nilpotent.

It was this observation that was the key to the development of the theory of [Mzk1-4].

(B.) The Characteristic p Theory

Let p be an *odd* prime. Let $\mathcal{N}_{g,r} \subseteq (\mathcal{S}_{g,r})_{\mathbf{F}_p}$ denote the closed algebraic substack of indigenous bundles with square nilpotent p-curvature. Then one has the following key result ([Mzk1], Chapter II, Theorem 2.3):

Theorem 3. *The natural map $\mathcal{N}_{g,r} \rightarrow (\mathcal{M}_{g,r})_{\mathbf{F}_p}$ is a finite, flat, local complete intersection morphism of degree p^{3g-3+r}. Thus, up to "isogeny" (i.e., up to the fact that this degree is not equal to one), $\mathcal{N}_{g,r}$ defines a canonical section of the Schwarz torsor $(\mathcal{S}_{g,r})_{\mathbf{F}_p} \rightarrow (\mathcal{M}_{g,r})_{\mathbf{F}_p}$ in characteristic p.*

It is this stack $\mathcal{N}_{g,r}$ of *nilcurves* – i.e., hyperbolic curves in characteristic p equipped with an indigenous bundle with square nilpotent p-curvature – which is the central object of study in the theory of [Mzk1-4].

Many facts are now known about the *finer structure of* $\mathcal{N}_{g,r}$. One interesting consequence of this structure theory of $\mathcal{N}_{g,r}$ is that *it gives a new proof of the connectedness of* $(\mathcal{M}_{g,r})_{\mathbf{F}_p}$ (for p large relative to g). This fact is interesting – relative to the claim that this theory is a p-adic version of Teichmüller theory – in that one of the first applications of classical complex Teichmüller theory is to prove the connectedness of $\mathcal{M}_{g,r}$. Also, it is interesting to note that F. Oort has succeeded in giving a proof of the connnectedness of the moduli stack of principally polarized abelian varieties by applying the structure theory of certain natural substacks of this moduli stack in characteristic p.

(C.) Canonical Liftings

So far, we have been discussing the characteristic p theory. Ultimately, however, we would like to know if the various characteristic p objects discussed in (B.) lift canonically to objects which are flat over $\mathbf{Z}_p$. Unfortunately, it seems that it is unlikely that $\mathcal{N}_{g,r}$ itself lifts canonically to some sort of natural $\mathbf{Z}_p$-flat object. If, however, we consider the open substack – called the *ordinary locus* – $(\mathcal{N}^{\mathrm{ord}}_{g,r})_{\mathbf{F}_p} \subseteq \mathcal{N}_{g,r}$ which is the étale locus of the morphism $\mathcal{N}_{g,r} \rightarrow (\mathcal{M}_{g,r})_{\mathbf{F}_p}$, then (since the étale site is invariant under nilpotent thickenings) we get a canonical lifting, i.e., an étale morphism

$$\mathcal{N}^{\mathrm{ord}}_{g,r} \rightarrow (\mathcal{M}_{g,r})_{\mathbf{Z}_p}$$

of p-adic formal stacks. Over $\mathcal{N}^{\mathrm{ord}}_{g,r}$, one has the sought-after canonical p-adic splitting of the Schwarz torsor (cf. Theorem 0.1 of the Introduction of [Mzk1]):

Theorem 4. *There is a canonical section $\mathcal{N}^{\mathrm{ord}}_{g,r} \rightarrow \mathcal{S}_{g,r}$ of the Schwarz torsor over $\mathcal{N}^{\mathrm{ord}}_{g,r}$ which is the unique section having the following property: There exists a lifting of Frobenius $\Phi_{\mathcal{N}} : \mathcal{N}^{\mathrm{ord}}_{g,r} \rightarrow \mathcal{N}^{\mathrm{ord}}_{g,r}$ such that the indigenous bundle on the tautological hyperbolic curve over $\mathcal{N}^{\mathrm{ord}}_{g,r}$ defined by the section $\mathcal{N}^{\mathrm{ord}}_{g,r} \rightarrow \mathcal{S}_{g,r}$ is* invariant *with respect to the Frobenius action defined by $\Phi_{\mathcal{N}}$.*

Moreover, it turns out that the *Frobenius lifting* $\Phi_{\mathcal{N}} : \mathcal{N}^{\text{ord}}_{g,r} \to \mathcal{N}^{\text{ord}}_{g,r}$ (i.e., morphism whose reduction modulo p is the Frobenius morphism) has the special property that $\frac{1}{p} \cdot \mathrm{d}\Phi_{\mathcal{N}}$ induces an isomorphism $\Phi^*_{\mathcal{N}}\Omega_{\mathcal{N}^{\text{ord}}_{g,r}} \cong \Omega_{\mathcal{N}^{\text{ord}}_{g,r}}$. Such a Frobenius lifting is called *ordinary.* It turns out that any ordinary Frobenius lifting (i.e., not just $\Phi_{\mathcal{N}}$) defines a set of *canonical multiplicative coordinates* in a formal neighborhood of any point α valued in an algebraically closed field k of characteristic p, as well as a *canonical lifting* of α to a point valued in $W(k)$ (Witt vectors with coefficients in k). Moreover, there is a certain analogy between this general theory of ordinary Frobenius liftings and the theory of *real analytic Kähler metrics* (which also define canonical coordinates). Relative to this analogy, the canonical Frobenius lifting $\Phi_{\mathcal{N}}$ on $\mathcal{N}^{\text{ord}}_{g,r}$ may be regarded as corresponding to the *Weil-Petersson metric* on complex Teichmüller space (a metric whose canonical coordinates are the coordinates arising from the Bers uniformization of Teichmüller space). Thus, $\Phi_{\mathcal{N}}$ is, in a very real sense, a p-adic analogue of the Bers uniformization in the complex case. Moreover, there is, in fact, a canonical ordinary Frobenius lifting on the "ordinary locus" of the tautological curve over $\mathcal{N}^{\text{ord}}_{g,r}$ whose relative canonical coordinate is analogous to the canonical coordinate arising from the Köbe uniformization of a hyperbolic curve.

Next, we observe that Serre-Tate theory for ordinary (principally polarized) abelian varieties (cf., e.g., the case of *ordinary elliptic curves* discussed in I., (C.)) may also be formulated as arising from a certain canonical ordinary Frobenius lifting. Thus, the Serre-Tate parameters (respectively, Serre-Tate canonical lifting) may be identified with the canonical multiplicative parameters (respectively, canonical lifting to the Witt vectors) of this Frobenius lifting. That is to say, *in a very concrete and rigorous sense, Theorem 4 may be regarded as the analogue of Serre-Tate theory for hyperbolic curves.* Nevertheless, we remark that it is *not* the case that the condition that a nilcurve be ordinary (i.e., defines a point of $(\mathcal{N}^{\text{ord}}_{g,r})_{\mathbf{F}_p} \subseteq \mathcal{N}_{g,r}$) either implies or is implied by the condition that its Jacobian be ordinary. Although this fact may disappoint some readers, it is in fact very natural when viewed relative to the general analogy between ordinary Frobenius liftings and real analytic Kähler metrics discussed above. Indeed, relative to this analogy, we see that it corresponds to the fact that, when one equips $\mathcal{M}_g$ with the Weil-Petersson metric and $\mathcal{A}_g$ (the moduli stack of principally polarized abelian varieties) with its natural metric arising from the Siegel upper half-plane uniformization, *the Torelli map $\mathcal{M}_g \to \mathcal{A}_g$ is not isometric.*

Finally, we remark that once one develops these theories of canonical liftings, one also gets accompanying canonical (crystalline) Galois representations of the arithmetic fundamental group of the tautological curve over $\mathcal{N}^{\text{ord}}_{g,r}$ (and its Lubin-Tate generalizations) into PGL_2 of various complicated rings with Galois action. It turns out that *these Galois representations are the analogues of the canonical representation $\rho_{\mathcal{X}}$* (of I., (E.)). Moreover, it is these Galois representations which exhibit p-adic Teichmüller theory as a *modular intrinsic Hodge theory*, i.e., an equivalence between (local) algebraic moduli and structures arising from the Galois action on the étale fundamental group of the tautological curve – cf. the discussion of I., (D.).

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